4·8. UTILITY FUNCTION

A function which denotes the measure of satisfaction or utility an individual gets from the consumption of the commodities per unit time is known as the *utility function*. For example, for two commodities X_1 and X_2, the utility function, $U = f(x_1, x_2)$, denotes the level of satisfaction (utility) which the individual gets on consuming the quantities x_1 and x_2 of the commodities X_1 and X_2 respectively.

The *marginal utility* of the commodity X_i, $(i = 1,2)$, is defined as the approximate change in the total utility resulting from one-unit change in the consumption of the commodity X_i, per unit time.

$$\text{Marginal utility of } X_i = MU4x_1 = \frac{\partial f}{\partial x_i}\ ; i = 1, 2 \qquad \ldots (4\cdot55)$$

The rate of change of the marginal utility of X_i is given by $\frac{\partial}{\partial x_i}(MU_{x_i})$; $i = 1, 2$.

(*i*) If $\frac{\partial}{\partial x_i}(MU_{x_i}) > 0$, then MU_{x_i} is an increasing function of x_i, *i.e.*, the marginal utility of x_i increases as x_i increases.

(*ii*) If $\frac{\partial}{\partial x_i}(MU_{x_i}) < 0$, then MU_{x_i} is a decreasing function of x_i, *i.e.*, MU_{x_i} decreases as x_i increases.

4·8·1. Constrained Utility Maximisation. Our objective is to determine the values of x_1 and x_2 of the quantities consumed of the two commodities so as to maximise the total utility or satisfaction level. Hypothetically, the consumer can purchase infinite quantities of both the commodities. However, in reality this assumption has no relevance because the quantities of the commodities purchased depend on the purchasing power (income) of the consumer.

Hence our problem is to maximise the utility function :

$$U = f(x_1, x_2) \qquad \text{... (4·56)}$$

for variations in x_1 and x_2 subject to the constraint that his purchasing power (income) is given (fixed).

If p_1 and p_2 are the prices per unit and, x_1 and x_2 are the amounts of the quantities consumed, of the commodities X_1 and X_2 respectively, then the *budget constraint* is :

$$Y_o = p_1 x_1 + p_2 x_2 \quad \text{or} \quad p_1 x_1 + p_2 x_2 - y_0 = 0 \qquad \text{...(4·57)}$$

where

y_0 = Income of the amount of money to be spent on both the commodities

$p_1 x_1$ = Amount of money spent on commodity X_1

$p_2 x_2$ = Amount of money spent on commodity X_2

We want to maximise the utility function in (4·56) subject to the constraint (4·57), for variations in x_1 and x_2.

Resulting (4·57), we get
$$x_2 = \frac{y_0 - p_1 x_1}{p_2} \qquad \text{...(4·58)}$$

Substituting this value of x_2 in (4·56), we get

$$U = f\left(x_1, \frac{y_0 - p_1 x_1}{p_2}\right) = \phi(x_1), \text{ (say)}, \qquad \text{... (4·59)}$$

where $\phi(x_1)$ is a function of single variable x_1 only.

The maximum of $U = \phi(x_1)$, for variations in x_1, is the solution of the equations :

$$\frac{du}{dx_1} = \frac{d}{dx_1}[\phi(x_1)] = 0 \qquad \text{...(4·60)}$$

and
$$\frac{d^2u}{dx_1^2} = \frac{d^2}{dx_1^2}[\phi(x_1)] < 0$$

Substituting the value of x_1 so obtained in (4·58), we get x_2.

Finally, substituting these values of x_1 and x_2 in (4·56), we get a measure of the maximum satisfaction (utility) which the consumer gets from the consumption of the commodities X_1 and X_2.

Necessary Condition for Maximisation of $u = f(x_1, x_2)$ subject to Budget Constraint (4· 57).

Taking total differential in (4·56), we get

$$du = \frac{\partial f}{\partial x_1}\cdot dx_1 + \frac{\partial f}{\partial x_2}\cdot dx_2 \quad \Rightarrow \quad \frac{du}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dx_1}$$

$$\therefore \quad \frac{du}{dx_1} = \frac{\partial f}{\partial x_1} - \frac{p_1}{p_2}\cdot\frac{\partial f}{\partial x_2}\left[\because \frac{dx_2}{dx_1} = -\frac{p_2}{p_1} \text{ From (4·56)}\right] \qquad \text{...(4·61)}$$

For maximum of u, we have

$$\frac{du}{dx_1} = 0 \quad \Rightarrow \quad f_{x_1} - \frac{p_1}{p_2} f_{x_2} = 0$$

$$\Rightarrow \quad \frac{f_{x_1}}{f_{x_2}} = \frac{p_1}{p_2} \quad \Rightarrow \quad \frac{MU_{x_1}}{MU_{x_2}} = \frac{p_1}{p_2} \qquad \text{...(4·62)}$$

i.e., for optimum utility, the ratio of the marginal utilities must equal the ratio of the prices of the commodities, or equivalently :

$$\frac{MU_{x_1}}{p_1} = \frac{MU_{x_2}}{p_2} \qquad \text{...(4·62 a)}$$

i.e., the marginal utility divided by the price of the commodity must be same for both the commodities.

Sufficient Condition. The condition in (4·62) or (4·62 *a*) is only a necessary condition and not a sufficient condition for the maximum value of u. In fact, these are the conditions for an extremum (maximum or minimum) of $U = f(x_1, x_2)$. To obtain the sufficient condition for maximum of U, we have to find the second order derivative of U in (4·59). The sufficient condition for maximum of U is that

$$\frac{d^2 u}{dx_1^2} < 0. \qquad \ldots(4 \cdot 63)$$